How to Build Regueiro's Polyhedra

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Abstract

Regueiro's polyhedra are constructions – with triangular faces - formed by trirectangular tetrahedrons and/or tetrahedrons with two orthogonal dihedral connected by common faces and, therefore, the sums of the squares of the areas of some of its faces is equal to the sums of the squares of the areas of the remaining triangular faces of the polyhedron. We will see how are they constructed through the original hypothesis and the how the thesis is checked.

The Ancient Origins

In 1965 my vocation for mathematics awoke when finding a generalization of the theorem of Pythagoras in n dimensions. On the 50th anniversary of my first theorem, it is a good time to return to it as, although years later I got a simple demonstration to n dimensions published in 1981 [1] , I think they have failed to see the possibilities that opens this theorem.

In this case I will focus on constructing Regueiro's polyhedral with tetrahedrons with two orthogonal dihedral, which holds that $A_1^2+A_2^2=A_3^2+A_4^2$ [2], being the areas on either side of a dihedral the ones in each member of the equality equation. When applying repeatedly previous theorem, we will get a slew of new theorems and created figures.

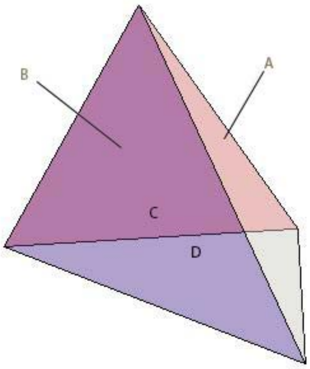
The news theorems and artworks

The artworks of *Mathematical Beauty* that I present in this edition of Bridges, illustrate these theorems. Let's see how do they work:
A tetrahedron with two orthogonal dihedral and no common vertex holds that $A_1^2+A_2^2=A_3^2+A_4^2$ But if in the face 4 we get over it another tetrahedron that holds $A_4^2+A_5^2=A_6^2+A_7^2$, then the resulting figure will hold $A_1^2+A_2^2+A_5^2=A_3^2+A_6^2+A_7^2$. Or if we add to the face 4 a trirectangular tetrahedron, $A_1^2+A_2^2=A_3^2+A_5^2+A_6^2+A_7^2$.
These are the type of theorems that verify the mentioned artworks.

The question that we will answer is how can we create, given two 3-dimensional vectors, a Regueiro's tetrahedron (with 2 orthogonal dihedral) and by extension, a Regueiro's polyhedron?

For example, we want to see how to build and demonstrate that they form a Regueiro's tetrahedron the following triplets of points, with the origin of coordinates, make the tetrahedron:

(1,0,1), (2,0,0), (2,3,0)// (1,2,3), (2,2,1), (2,0,1)// (4, 0, 0), (1, -Sqrt[2], 1), (2, 0, 2)// (4, 0, 0), (3, 2, 0), (2, 4/3, - (4 Sqrt[2])/3))// (6, 0, 0), (2, 3, 0), (2/3, 1, -Sqrt[3])// (2, 0, 4), (4, 1, 0), (2, 3, 0)// (12,6,9), (6,6,3), (6,4,1)// (6,0,0),(12,3,0),(-2, -1/2, -Sqrt[7]/2). All tetrahedrons before are Regueiro's Polyhedra (each three vectors)



A tetrahedron with two orthogonal dihedral verify $\mathbf{A^2+B^2=C^2+D^2}$ **[2] Regueiro's Theorem**

How to build

Using *Mathematica*, we can calculate the third vector (**t1**), given the initial vectors **v1** and **v2**, to build a tetrahedron with two orthogonal dihedrals.
Cross [v1 , v2] .Cross [t1 , v2] == 0 and Cross [v1 - t1, v1] .Cross [v1 - t1, v1 - v2] == 0 , that is, it has two orthogonal dihedral without a common vertex. Or verify that Regueiro's theorem is true in the examples cited above.

Norm[Cross[v1, v2]]^2 + Norm[Cross[t1, v2]]^2= Norm[Cross[v1 - t1, v1]]^2 + Norm[Cross[v1 - t1, v1 - v2]]^2 (*Mathematica* notation)

Given **v1** and **v2**, the process I use to calculate **t1**, the third vector is as follows:
Assign a value to one of the three incognita to facilitate calculations as, if you do not make it this way, enormous calculations will take place in several pages.

Value **t1** = (2, y, z), (for example)
So, we solve as follows:
Solve[{ Cross[v1, v2].Cross[t1, v2]== 0, Cross[v1 - t1, v1].Cross[v1 - t1, v1 - v2] == 0}, {y, z}]

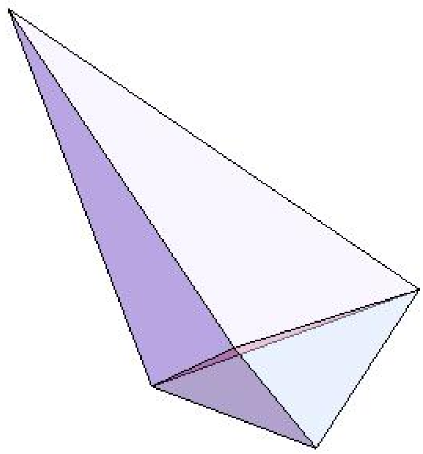
Values *y* and *z* that make the tetrahedron to have 2 orthogonal dihedral. This process is constructive, it allows us to take one of the new calculated faces (or previous ones) as the origin for new calculations (e.g., **v1** and **t1**) that will give us a new Regueiro's tetrahedron, but with a face in common with the first one, and so on , we can grow the geometric body adding tetrahedra. This is the origin of the artistic artwork that I present in *Bridges*. Besides, occasionally, we can insert trirectangular trihedrons that match on one side (usually the base) of the already obtained figure.

For instance, we start with **v1** = {4, -2, 3} and **v2** = {2, 2, 1}, and we want to calculate the third vector, **t1**
t1={2, y, z} of the tetrahedron.
Solve [{Cross[v1, v2].Cross[t1, v2]==0,Cross[v1 - t1, v1].Cross[v1 - t1, v1 - v2]==0}, {y, z}]

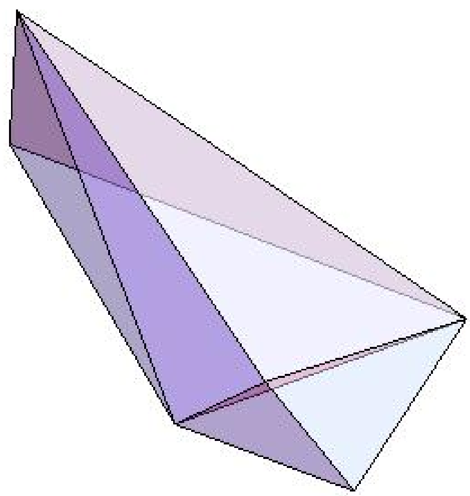
If now we assume the vectors **v1** = {2, 2 , 1} and **v2** = {2, 18/19 -13/19 } and build the third vector **t1**, we will get a figure that meet $A_1^2+A_2^2+A_5^2=A_3^2+A_6^2+A_7^2$. etc.
Summing up, like in the polyhedrons of *Mathematical Beauty*, from the Art Exhibition in Bridges 2015, we will see which quadratic formulas can be applied if we construct the polyhedron with special careful.

In fact, if we take pts = {{0, 0, 0}, {2, 0, 0}, {1, 0, 1}, {2, 3, 0}, {-3, 2, 3}, {-3.4064667279067278`, 2.270977818604485`, 1}, {0.5, 1.5126690402164342`, -1.7320044732678939`}} we can test that the following tetrahedrons verify $A^2+B^2=C^2+D^2$: {2, 0, 0}, {1, 0, 1}, {2, 3, 0} (well, it's best to check, (with Norm...) in this order,{1, 0, 1}, {2, 0, 0}, {2, 3, 0}, because the test it's not commutative respect to the values of v1 and v2). After we check for {1, 0, 1}, {2, 3, 0},{-3, 2, 3} and for {2, 3, 0},{-3, 2, 3}, {-3.4064667279067278, 2.270977818604485, 1}. And finally for {2, 3, 0}, {-3.4064667279067278, 2.270977818604485, 1}, {0.5, 1.5126690402164342, -1.7320044732678939} the four sets of vectors verify $A^2+B^2=C^2+D^2$. Then we can produce the images of the sequence with the order Graphics3D [{Opacity@0.4 ,Table[Tetrahedron [pts [[i]]], { i , ti }]} , Boxed -> False] if we take ti = {{1, 2, 3, 4}, {1, 3, 4, 5}, {1, 4, 5, 6}, {1, 4, 6, 7}}

The implicit algorithm is: given three vectors u, v, w that create a Regueiro's polyhedra, we construct over the face of vectors u, v a new polyhedron of this type. So, over the face the vectors v, w or u, w, or w-u and v-u. What is the formula in this case? the faces A, B, C, and D disappear and the result is something like $A_1^2+A_2^2+A_3^2+A_4^2+A_5^2+A_6^2=A_7^2+A_8^2+A_9^2+A_{10}^2+A_{11}^2+A_{12}^2$. Naturally, we can change the algorithm – at any moment, at any step- and build over a face of a new tetrahedron. In the above example, the figures step by step are

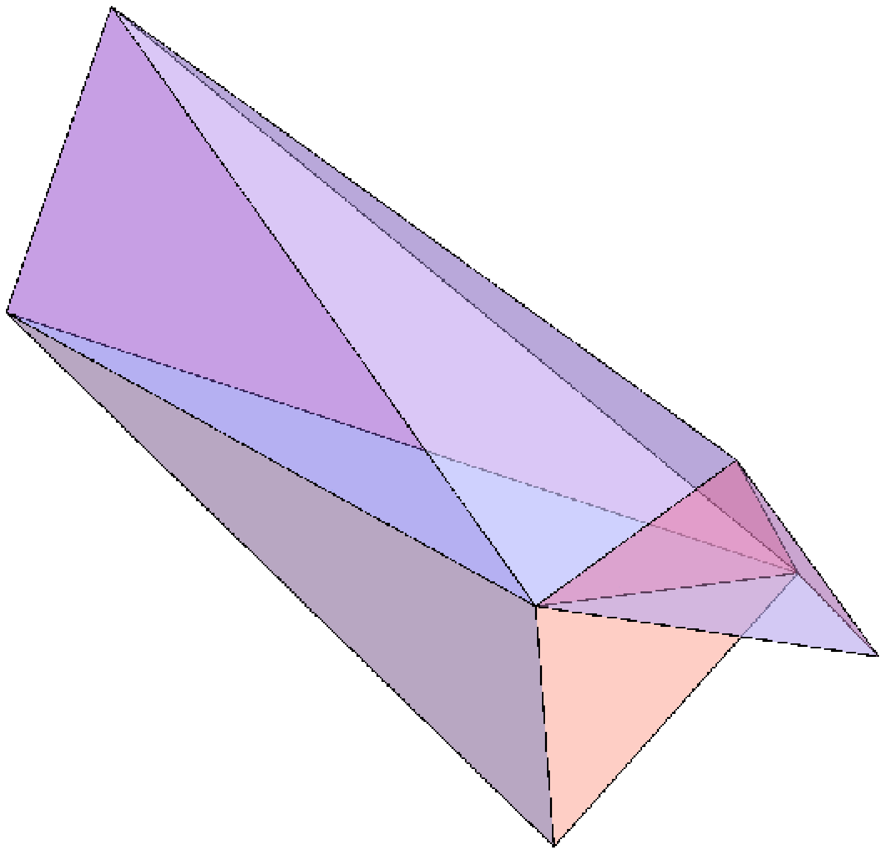


with ti = {{1, 2, 3, 4}, {1, 3, 4, 5}]



with ti = {{1, 2, 3, 4}, {1, 3, 4, 5}, {1, 4, 5, 6}}

And finally



with ti = {{1, 2, 3, 4}, {1, 3, 4, 5}, {1, 4, 5, 6}, {1, 4, 6, 7}}

The utility of a figure of this kind goes beyond mathematical beauty. Thus, in Optics it is necessary that certain beams change their direction, as it happens in the periscope straight prisms, and this figure is changed in an extraordinary and complex way. Because of this feature the usability increases in art.

References

[1] Manuel Díaz Regueiro. *Geometría métrica en un simplex de* R^n. Gaceta Matemática. Tomo XIII. Número 5-6. Madrid. 1981.

[2] Manuel Díaz Regueiro. *Geometría métrica en tetraedro*. Boletín das Ciencias. Santiago. 1989. The two articles are in the web allegue.com/artigos.